

Quantum state transfer for multi-input linear quantum systems

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Abstract—Effective state transfer is one of the most important problems in quantum information processing. Typically, a quantum information device is composed of many subsystems with multi-input ports. In this paper, we develop a general theory describing the condition for perfect state transfer from the multi-input ports to the internal system components, for general passive linear quantum systems. The key notion used is the zero of the transfer function matrix. Application to entanglement generation and distribution in a quantum network is also discussed.

I. INTRODUCTION

A quantum state transducer, that for instance transfers an optical state to a solid state, is an indispensable component contained in various types of quantum information processors. For instance, such a state transfer procedure is involved in every quantum memory architecture [1], [2], [3], [4], which is typically used for building a quantum repeater in quantum communication networks [5]. Towards a systematic and effective design of state transfer protocol, in [6] two of the authors developed a general theory for single-input and single-output (SISO) passive linear quantum systems [7], [8]; the result obtained is that the input state encoded in an appropriately shaped wave function can be perfectly transferred to the system. A remarkable fact is that such a wave function can be completely characterized in terms of the zeros of the transfer function of the system, which thus revealed a close connection between systems and control theory and the important task in quantum information science.

Based on this background, in this paper, we aim to extend the result of [6] to the case of multi-input and multi-output (MIMO) general linear passive systems. In fact the memory systems studied in the above-referred papers [1], [2], [3] are all MIMO systems. Also a hybridized system conducting frequency conversion between e.g. an optical cavity and a microwave circuit is essentially an MIMO system [9], [10], [11]. On the other hand, it is well known in classical systems and control theory [12] that extending the notion of zeros from the SISO case to the MIMO case is quite nontrivial. This is because in the MIMO case we are dealing with a transfer function *matrix*, and the zeros of this matrix can have several definitions; for instance, a *transmission zero* is

defined as a complex number at which the rank of the transfer function matrix drops, while a *blocking zero* is a complex number at which the transfer function matrix becomes a zero matrix. Therefore, the goal of this paper is to deduce the condition for perfect state transfer and how that condition can be characterized by the zeros of the transfer function matrix.

Notation: for a matrix $A = (a_{ij})$, the symbols A^\dagger , A^\top , and $A^\#$ represent its Hermitian conjugate, transpose, and complex conjugation in elements of A , respectively; i.e., $A^\dagger = (a_{ji}^*)$, $A^\top = (a_{ji})$, and $A^\# = (a_{ij}^*)$. For a matrix of operators we use the same notation, in which case a_{ij}^* denotes the adjoint to a_{ij} .

II. PRELIMINARIES

A. Model of the system and input

Let us consider the following MIMO passive linear quantum system [7], [8]:

$$\frac{d\mathbf{a}}{dt} = A\mathbf{a} - C^\dagger S\mathbf{b}, \quad \tilde{\mathbf{b}} = C\mathbf{a} + S\mathbf{b}. \quad (1)$$

Here $\mathbf{a} = [a_1, \dots, a_n]^\top$ is the vector of system annihilation operators. This system has m input channels represented by the vector of field annihilation operators $\mathbf{b} = [b_1, \dots, b_m]^\top$, and $\tilde{\mathbf{b}}$ is the corresponding output. These are infinite dimensional operators satisfying e.g. $a_i(t)a_j^*(t) - a_j^*(t)a_i(t) = \delta_{ij} \forall t$ and $b_i(t)b_j^*(t') - b_j^*(t')b_i(t) = \delta_{ij}\delta(t-t') \forall t, t'$. In the dynamical equation, $C \in \mathbb{C}^{m \times n}$ represents the system-field coupling; also $A = -i\Omega - C^\dagger C/2$, where the $n \times n$ Hermitian matrix $\Omega = \Omega^\dagger$ is related to the system Hamiltonian. Finally S is a $m \times m$ unitary matrix, representing the scattering process of \mathbf{b} .

In the state transfer problem considered in this paper, we assume that the input is given by a continuous-mode single-photon field state. This state is defined in terms of the following annihilation and creation process operators:

$$B(\xi) = \int_{-\infty}^{\infty} \xi^*(t)b(t)dt, \quad B^*(\xi) = \int_{-\infty}^{\infty} \xi(t)b^*(t)dt. \quad (2)$$

$\xi(t)$ is an associated function in \mathbb{C} , representing the shape of the optical pulse field. Also $\xi(t)$ satisfies the normalization condition $\int_{-\infty}^{\infty} |\xi(t)|^2 dt = 1$. Due to this, $B(\xi)$ and $B^*(\xi)$ satisfy the relation $B(\xi)B^*(\xi) - B^*(\xi)B(\xi) = 1$. The continuous-mode single photon field state is produced by acting $B^*(\xi)$ on the vacuum field $|0\rangle_f$ as follows [13], [14]:

$$|1_\xi\rangle_f = B^*(\xi)|0\rangle_f = \int_{-\infty}^{\infty} \xi(t)b^*(t)dt|0\rangle_f. \quad (3)$$

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This is the continuous-mode version of the single-mode single-photon state $|1\rangle = a^*|0\rangle$ where a^* is the single-mode creation operator and $|0\rangle$ is the ground state. Note that ${}_f\langle 1_\xi|1_\xi\rangle_f = 1$ holds due to the normalization condition of $\xi(t)$. Also from the relation ${}_f\langle 1_\xi|b^*(t)b(t)|1_\xi\rangle_f = |\xi(t)|^2$, $\xi(t)$ has the meaning of the wave function such that $|\xi(t)|^2$ represents the probability of photo detection per unit time.

B. Zeros of a passive linear system

The transfer function matrix of the system (1) is given by

$$G(s) = [I - C(sI - A)^{-1}C^\dagger]S.$$

Here we give two definitions of zeros of a general $m \times m$ transfer function matrix $G(s)$ [12].

Definition 1: If there exist $z \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$ such that $G(z)\mathbf{u} = 0$, then z is called a *transmission zero*.

Definition 2: If there exists $z \in \mathbb{C}$ such that $G(z) = 0$, then z is called a *blocking zero*.

The following three facts are used in this paper.

Fact 1: Suppose that $z \in \mathbb{C}$ is a (blocking or transmission) zero of $G(s)$, and it is not a pole of $G(s)$. Then z is an eigenvalue of $-A^\dagger$.

Proof: Let us first consider the case of a blocking zero. This means there exists $z \in \mathbb{C}$ such that $[I - C(zI - A)^{-1}C^\dagger]S = 0$. Now let us define $V := (zI - A)^{-1}C^\dagger S$; then we have $(zI - A)VS^\dagger = C^\dagger$ and $CV = S$. These two equations lead to $(zI - A)V = C^\dagger CV$ and therefore $(A + C^\dagger C)V = zV$. Thus $-A^\dagger V = zV$; note that z is degenerated in the eigenspace $\text{span}(V)$.

The case of transmission zero is almost the same. The definition is that there exist $z \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$ such that $G(z)\mathbf{u} = [I - C(zI - A)^{-1}C^\dagger]S\mathbf{u} = 0$. Again define $V := (zI - A)^{-1}C^\dagger S$, which leads to $(zI - A)VS^\dagger = C^\dagger$ and $CV\mathbf{u} = S\mathbf{u}$; hence we have $(zI - A)V\mathbf{u} = C^\dagger CV\mathbf{u}$ and $-A^\dagger V\mathbf{u} = zV\mathbf{u}$. Note from this the transmission-zero vector \mathbf{u} and the eigenvector of $-A^\dagger$, \mathbf{v} , are connected by $\mathbf{v} = V\mathbf{u}$. This further yields $S^\dagger C\mathbf{v} = S^\dagger CV\mathbf{u} = S^\dagger S\mathbf{u} = \mathbf{u}$, hence $\mathbf{u} = S^\dagger C\mathbf{v}$. \square

Fact 2: If A is Hurwitz, then all (blocking or transmission) zeros of $G(s)$ are unstable zeros.

Proof: Let λ be an eigenvalue of A , i.e. $\det(\lambda I - A) = 0$. This yields $\det(\lambda^* I - A^\dagger) = 0$. Then from Fact 1, a zero of $G(s)$, z , is given by $z = -\lambda^*$. Hence $\text{Re}(z) = -\text{Re}(\lambda) > 0$. \square

Fact 3: If A is Hurwitz, $G(s)$ always has a transmission zero.

Proof: We begin with the eigen-equation $-A^\dagger \mathbf{v} = z\mathbf{v}$. Then from $-A^\dagger = A + C^\dagger C$, we have $(A + C^\dagger C)\mathbf{v} = z\mathbf{v}$. Now from Fact 2, z is not an eigenvalue of A , hence $(z - A)^{-1}$ always exists; thus we have $\mathbf{v} = (z - A)^{-1}C^\dagger C\mathbf{v}$. This yields $C\mathbf{v} = C(z - A)^{-1}C^\dagger C\mathbf{v}$ and further $[I - C(z - A)^{-1}C^\dagger]C\mathbf{v} = 0$. Therefore we end up with $G(z)S^\dagger C\mathbf{v} = 0$, meaning that there always exists a transmission zero. Note again we find the transmission-zero vector \mathbf{u} and the eigenvector \mathbf{v} are connected by $\mathbf{u} = S^\dagger C\mathbf{v}$. \square

III. GENERAL MIMO STATE TRANSFER

In what follows we assume that A is Hurwitz. Then the solution of the dynamics is given by

$$\mathbf{a}^\#(0) = U^* \mathbf{a}^\#(t_0) U = - \int_{t_0}^0 e^{-A^\dagger t} C^\top S^\# \mathbf{b}^\#(t) dt, \quad (4)$$

where U is the unitary operator describing the joint time evolution of the system and the field from the initial time t_0 to the final time 0. In particular the initial time is assumed to be $t_0 \rightarrow -\infty$. Let us define the matrix of functions

$$\begin{aligned} \Xi(t) &= \begin{bmatrix} \xi_{1,1}(t) & \cdots & \xi_{1,m}(t) \\ \vdots & & \vdots \\ \xi_{n,1}(t) & \cdots & \xi_{n,m}(t) \end{bmatrix} \\ &:= -e^{-A^\dagger t} C^\top S^\# \Theta(-t). \end{aligned} \quad (5)$$

$\Theta(-t)$ is the Heaviside step function taking 1 for $t \leq 0$ and 0 for $t > 0$. This matrix satisfies $\int_{-\infty}^{\infty} \Xi(t) \Xi(t)^\dagger dt = I$. Then we find

$$\begin{aligned} U^* \mathbf{a}^\#(t_0) U &= \begin{bmatrix} U^* a_1^*(t_0) U \\ \vdots \\ U^* a_n^*(t_0) U \end{bmatrix} = \int_{t_0}^0 \Xi(t) \mathbf{b}^\#(t) dt \\ &= \begin{bmatrix} B_1^*(\xi_{1,1}) + \cdots + B_m^*(\xi_{1,m}) \\ \vdots \\ B_1^*(\xi_{n,1}) + \cdots + B_m^*(\xi_{n,m}) \end{bmatrix}, \end{aligned} \quad (6)$$

where $B_k^*(\xi_{i,j})$ is the continuous-mode creation process operator on the k th input channel, defined by Eq. (2). This means that a special class of input field state can be perfectly transferred to the system. For instance let us consider the following entangled single-photon field state:

$$\begin{aligned} |\Psi(t_0)\rangle_f &= |1_{\xi_{1,1}}^{(1)}\rangle_f + \cdots + |1_{\xi_{1,m}}^{(m)}\rangle_f \\ &= |1_{\xi_{1,1}}, 0, \dots, 0\rangle_f + \cdots + |0, \dots, 0, 1_{\xi_{1,m}}\rangle_f \\ &= [B_1^*(\xi_{1,1}) + \cdots + B_m^*(\xi_{1,m})] |0, \dots, 0\rangle_f, \end{aligned}$$

where the definition (3) is used. Also we define $|1^{(j)}\rangle = |0, \dots, 1, \dots, 0\rangle$ with 1 appearing only in the j th component. Note ${}_f\langle \Psi(t_0) | \Psi(t_0) \rangle_f = 1$ due to $\int_{-\infty}^{\infty} \Xi(t) \Xi(t)^\dagger dt = I$. In this case, from Eq. (6), the final state of the whole system is calculated as

$$\begin{aligned} |\Psi(0)\rangle &= U |0, \dots, 0\rangle_s |\Psi(t_0)\rangle_f \\ &= U [B_1^*(\xi_{1,1}) + \cdots + B_m^*(\xi_{1,m})] |0, \dots, 0\rangle_s |0, \dots, 0\rangle_f \\ &= U [B_1^*(\xi_{1,1}) + \cdots + B_m^*(\xi_{1,m})] U^* |0, \dots, 0\rangle_s |0, \dots, 0\rangle_f \\ &= a_1^*(t_0) |0, \dots, 0\rangle_s |0, \dots, 0\rangle_f = |1, 0, \dots, 0\rangle_s |0, \dots, 0\rangle_f \\ &= |1^{(1)}\rangle_s |0, \dots, 0\rangle_f. \end{aligned}$$

This equation shows that the first mode of the system acquires the single photon from the field; i.e. perfect state transfer is realized.

More generally, if the input field state is given by

$$\begin{aligned} |\Psi(t_0)\rangle_f &= x_1 \left(|1_{\xi_{1,1}}^{(1)}\rangle_f + \cdots + |1_{\xi_{1,m}}^{(m)}\rangle_f \right) \\ &\quad + \cdots + x_n \left(|1_{\xi_{n,1}}^{(1)}\rangle_f + \cdots + |1_{\xi_{n,m}}^{(m)}\rangle_f \right) \\ &= |1_{\xi'_1}^{(1)}\rangle_f + \cdots + |1_{\xi'_m}^{(m)}\rangle_f, \end{aligned} \quad (7)$$

where $\xi'_j = x_1 \xi_{1,j} + \cdots + x_n \xi_{n,j}$ and $x_j \in \mathbb{C}$ an arbitrary coefficient satisfying $\sum_{j=1}^n |x_j|^2 = 1$, in this case the final state is

$$|\Psi(0)\rangle = \left(x_1 |1^{(1)}\rangle_s + \cdots + x_n |1^{(n)}\rangle_s \right) |0, \dots, 0\rangle_f. \quad (8)$$

Note that the pulse functions $\xi_{i,j}(t)$ do not depend on the (unknown) coefficients $\{x_j\}$; hence, if the single photon field state with classical information $\{x_j\}$ can be prepared, which is a challenging task experimentally, then it can be perfectly transferred to the system.

Example 1: Let us consider the case where the system is composed of two single-mode SISO subsystems specified by the system parameters (A_1, C_1) and (A_2, C_2) . (Thus A_1, A_2, C_1, C_2 are scalars.) These two subsystems can be placed at a distant location from a source. The two input fields are combined at a beam splitter before being sent to the two subsystems. Thus the whole system are specified by

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}, \quad S = \begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix}.$$

Here α and β represent the transmissivity and the reflectivity of the beam splitter, respectively, which are assumed to be real without loss of generality. Then we have

$$\begin{aligned} \Xi(t) &= \begin{bmatrix} \xi_{1,1}(t) & \xi_{1,2}(t) \\ \xi_{2,1}(t) & \xi_{2,2}(t) \end{bmatrix} \\ &= \begin{bmatrix} -\alpha e^{-A_1^* t} C_1 & -\beta e^{-A_1^* t} C_1 \\ -\beta e^{-A_2^* t} C_2 & -\alpha e^{-A_2^* t} C_2 \end{bmatrix}, \end{aligned} \quad (9)$$

and

$$\begin{bmatrix} U^* a_1^*(t_0) U \\ U^* a_2^*(t_0) U \end{bmatrix} = \begin{bmatrix} B_1^*(\xi_{1,1}) + B_2^*(\xi_{1,2}) \\ B_1^*(\xi_{2,1}) + B_2^*(\xi_{2,2}) \end{bmatrix}.$$

Then if the initial field state is prepared as

$$\begin{aligned} |\Psi(t_0)\rangle_f &= x_1 \left(|1_{\xi_{1,1}}, 0\rangle_f + |0, 1_{\xi_{1,2}}\rangle_f \right) \\ &\quad + x_2 \left(|1_{\xi_{2,1}}, 0\rangle_f + |0, 1_{\xi_{2,2}}\rangle_f \right) \\ &= |1_{x_1 \xi_{1,1} + x_2 \xi_{2,1}}, 0\rangle_f + |0, 1_{x_1 \xi_{1,2} + x_2 \xi_{2,2}}\rangle_f, \end{aligned} \quad (10)$$

the final system state is given by

$$|\Psi(0)\rangle_s = x_1 |1, 0\rangle_s + x_2 |0, 1\rangle_s.$$

That is, the two separately placed two subsystems are entangled. Note however that, to achieve the perfect state transfer, in general, the initial field state has to be entangled between the two input channels even before entering into the beam splitter. \square

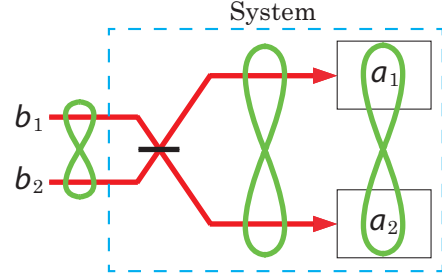


Fig. 1. Perfect state transfer from the field single-photon entangled state to the system single-photon entangled state.

IV. PERFECT STATE TRANSFER AND ZEROS

In the previous section we found that, in the general setup, the field state (7) can be perfectly transferred to the system state (8). That is, although engineering the entangled single-photon state (7) is challenging in experiment, perfect state transfer is in principle *always* possible. This can be understood in terms of systems and control theory as follows. In general, for a linear system if the input is of the form $\mathbf{u}(t) = \mathbf{u}e^{zt}$ with z a transmission zero and \mathbf{u} the corresponding transmission-zero vector, then the output is given by $\mathbf{y}(t) = G(z)\mathbf{u}e^{zt} = 0$ for all $t \leq 0$. That is, if the system has a transmission (or more strongly blocking) zero, then an appropriately chosen input can make the output always zero. The point here is that the passive linear quantum system always has a transmission zero as shown in Fact 3, and this is the reason why the field state (7) is perfectly absorbed into the system. Thus the questions arising here are how the pulse function can be represented in terms of zeros of the system, and what field state represented in terms of zeros can be perfectly transferred.

To answer these questions let us recall Facts 1 and 3. That is, the system always has a transmission zero z satisfying $G(z)\mathbf{u} = 0$ with \mathbf{u} the corresponding transmission-zero vector, and this satisfies the eigen-equation $-A^\dagger \mathbf{v} = z\mathbf{v}$ with \mathbf{v} the corresponding eigenvector. Further, \mathbf{u} and \mathbf{v} are related as $\mathbf{u} = S^\dagger C\mathbf{v}$. Therefore, from Eq. (4) we have

$$\begin{aligned} \mathbf{v}^\top \mathbf{a}^\#(0) &= - \int_{t_0}^0 \mathbf{v}^\top e^{-A^\dagger t} C^\top S^\dagger \mathbf{b}^\#(t) dt \\ &= - \int_{t_0}^0 e^{zt} \mathbf{v}^\top C^\top S^\dagger \mathbf{b}^\#(t) dt \\ &= - \int_{t_0}^0 e^{zt} \mathbf{u}^\top \mathbf{b}^\#(t) dt \\ &= \mathbf{u}^\top \mathbf{B}^\#(-e^{zt}) \\ &= u_1 B_1^*(-e^{zt}) + \cdots + u_m B_m^*(-e^{zt}), \end{aligned} \quad (11)$$

where we have defined

$$\mathbf{B}^\#(\xi) = [B_1^*(\xi), \dots, B_m^*(\xi)]^\top.$$

Note that $-e^{zt}$ is not normalized, but let us keep this unnormalized pulse function to explicitly see the transmission

zero z . We here chose the following field input state:

$$\begin{aligned} |\Psi(t_0)\rangle_f &= \mathbf{u}^\top \mathbf{B}^\#(-e^{zt})|0, 0, \dots, 0\rangle_f \\ &= \left[u_1 B_1^*(-e^{zt}) + \dots + u_m B_m^*(-e^{zt}) \right] |0, 0, \dots, 0\rangle_f \\ &= u_1 |1_{-e^{zt}}^{(1)}\rangle_f + \dots + u_m |1_{-e^{zt}}^{(m)}\rangle_f. \end{aligned}$$

Then the final system-field state is given by

$$\begin{aligned} |\Psi(0)\rangle &= U|0, \dots, 0\rangle_s |\Psi(t_0)\rangle_f \\ &= \mathbf{u}^\top U \mathbf{B}^\#(-e^{zt}) U^* |0, \dots, 0\rangle_s |0, \dots, 0\rangle_f \\ &= \mathbf{v}^\top \mathbf{a}^\#(t_0) |0, \dots, 0\rangle_s |0, \dots, 0\rangle_f \\ &= \left[v_1 |1^{(1)}\rangle_s + \dots + v_n |1^{(n)}\rangle_s \right] |0, \dots, 0\rangle_f. \end{aligned}$$

Hence certainly the input pulse function needs to be of the rising exponential form specified by the transmission zero z , in order to achieve the perfect state transfer. In particular, the input field state has coefficients specified by \mathbf{u} , and the final system state has coefficients specified by \mathbf{v} .

The generalization is straightforward. Let us consider the case where the system has \bar{m} ($\leq m$) transmission zeros, $z_1, \dots, z_{\bar{m}}$, with corresponding transmission-zero vectors $\mathbf{u}_1, \dots, \mathbf{u}_{\bar{m}}$. Then from Eq. (11) we have

$$\begin{aligned} \begin{bmatrix} \mathbf{v}_1^\top \mathbf{a}^\#(0) \\ \vdots \\ \mathbf{v}_{\bar{m}}^\top \mathbf{a}^\#(0) \end{bmatrix} &= \begin{bmatrix} \mathbf{u}_1^\top \mathbf{B}^\#(-e^{z_1 t}) \\ \vdots \\ \mathbf{u}_{\bar{m}}^\top \mathbf{B}^\#(-e^{z_{\bar{m}} t}) \end{bmatrix} \\ &= \begin{bmatrix} u_{1,1} B_1^*(-e^{z_1 t}) + \dots + u_{1,m} B_m^*(-e^{z_1 t}) \\ \vdots \\ u_{\bar{m},1} B_1^*(-e^{z_{\bar{m}} t}) + \dots + u_{\bar{m},m} B_m^*(-e^{z_{\bar{m}} t}) \end{bmatrix} \end{aligned} \quad (12)$$

where $\mathbf{u}_j = [u_{j,1}, \dots, u_{j,m}]^\top$. Now we set the field input state to be

$$\begin{aligned} |\Psi(t_0)\rangle_f &= \left[x_1 \mathbf{u}_1^\top \mathbf{B}^\#(-e^{z_1 t}) + \dots + x_{\bar{m}} \mathbf{u}_{\bar{m}}^\top \mathbf{B}^\#(-e^{z_{\bar{m}} t}) \right] \\ &\quad \times |0, 0, \dots, 0\rangle_f \\ &= \left[x_1 \left(u_{1,1} B_1^*(-e^{z_1 t}) + \dots + u_{1,m} B_m^*(-e^{z_1 t}) \right) + \dots \right. \\ &\quad \left. + x_{\bar{m}} \left(u_{\bar{m},1} B_1^*(-e^{z_{\bar{m}} t}) + \dots + u_{\bar{m},m} B_m^*(-e^{z_{\bar{m}} t}) \right) \right] \\ &\quad \times |0, 0, \dots, 0\rangle_f \\ &= \left[B_1^*(-x_1 u_{1,1} e^{z_1 t} - \dots - x_{\bar{m}} u_{\bar{m},1} e^{z_{\bar{m}} t}) + \dots \right. \\ &\quad \left. + B_m^*(-x_1 u_{1,m} e^{z_1 t} - \dots - x_{\bar{m}} u_{\bar{m},m} e^{z_{\bar{m}} t}) \right] \\ &\quad \times |0, 0, \dots, 0\rangle_f \\ &= \left[B_1^*(u'_1) + \dots + B_m^*(u'_m) \right] |0, 0, \dots, 0\rangle_f \\ &= |1_{u'_1}^{(1)}\rangle_f + \dots + |1_{u'_m}^{(m)}\rangle_f, \end{aligned} \quad (13)$$

where $x_1, \dots, x_{\bar{m}}$ are arbitrary coefficients and

$$\begin{aligned} \mathbf{u}'(t) &:= \begin{bmatrix} u'_1(t) \\ \vdots \\ u'_m(t) \end{bmatrix} \\ &= \begin{bmatrix} -x_1 u_{1,1} e^{z_1 t} - \dots - x_{\bar{m}} u_{\bar{m},1} e^{z_{\bar{m}} t} \\ \vdots \\ -x_1 u_{1,m} e^{z_1 t} - \dots - x_{\bar{m}} u_{\bar{m},m} e^{z_{\bar{m}} t} \end{bmatrix} \\ &= -x_1 \mathbf{u}_1 e^{z_1 t} - \dots - x_{\bar{m}} \mathbf{u}_{\bar{m}} e^{z_{\bar{m}} t}. \end{aligned} \quad (14)$$

Then by defining the vector

$$\mathbf{v}' := x_1 \mathbf{v}_1 + \dots + x_{\bar{m}} \mathbf{v}_{\bar{m}}, \quad (15)$$

we find that

$$\begin{aligned} \mathbf{v}'^\top \mathbf{a}^\#(0) &= [x_1 \mathbf{v}_1^\top + \dots + x_{\bar{m}} \mathbf{v}_{\bar{m}}^\top] \mathbf{a}^\#(0) \\ &= x_1 \mathbf{u}_1^\top \mathbf{B}^\#(-e^{z_1 t}) + \dots + x_{\bar{m}} \mathbf{u}_{\bar{m}}^\top \mathbf{B}^\#(-e^{z_{\bar{m}} t}) \end{aligned}$$

and thus the final system-field state is given by

$$\begin{aligned} |\Psi(0)\rangle &= U|0, \dots, 0\rangle_s |\Psi(t_0)\rangle_f \\ &= \left[x_1 \mathbf{u}_1^\top U \mathbf{B}^\#(-e^{z_1 t}) U^* + \dots \right. \\ &\quad \left. + x_{\bar{m}} \mathbf{u}_{\bar{m}}^\top U \mathbf{B}^\#(-e^{z_{\bar{m}} t}) U^* \right] |0, \dots, 0\rangle_s |0, \dots, 0\rangle_f \\ &= \mathbf{v}'^\top \mathbf{a}^\#(t_0) |0, \dots, 0\rangle_s |0, \dots, 0\rangle_f \\ &= \left[v'_1 |1^{(1)}\rangle_s + \dots + v'_{\bar{m}} |1^{(\bar{m})}\rangle_s \right] |0, \dots, 0\rangle_f. \end{aligned} \quad (16)$$

Summarizing, if the field input state is given by Eq. (13) with pulse functions (14), then it is perfectly transferred to the system state given by Eq. (16) with coefficient (15); again, $\{\mathbf{u}_j\}$ are the transmission-zero vectors and $\{\mathbf{v}_j\}$ are the corresponding eigenvectors of $-\mathbf{A}^\dagger$. Note that, if we *formally* input Eq. (14) to the associated classical system with transfer function matrix $G(s)$, then the corresponding formal output is given by

$$\mathbf{y} = -x_1 G(z_1) \mathbf{u}_1 e^{z_1 t} - \dots - x_{\bar{m}} G(z_{\bar{m}}) \mathbf{u}_{\bar{m}} e^{z_{\bar{m}} t} = 0.$$

However, this does not mean that the input field state can be set for instance to the separable one $|1_{u'_1}\rangle_f \otimes \dots \otimes |1_{u'_m}\rangle_f$; the input state we need to prepare is the entangled state (13).

Example 2 (continued from Example 1): The transfer function matrix is given by

$$\begin{aligned} G(s) &= \begin{bmatrix} G_1(s) & 0 \\ 0 & G_2(s) \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix} \\ &= \begin{bmatrix} \alpha G_1(s) & \beta G_1(s) \\ \beta G_2(s) & -\alpha G_2(s) \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} G_j(s) &= 1 - C_j(s - A_j)^{-1} C_j^\dagger = 1 - \frac{|C_j|^2}{s - A_j} \\ &= \frac{s - A_j - |C_j|^2}{s - A_j}. \end{aligned}$$

Again note that (A_j, C_j) are scalars. Clearly $G_j(s)$ has a zero $z_j = A_j + |C_j|^2$. Here we assume that the two subsystems are different and as a result they have two different

zeros, i.e. $z_1 \neq z_2$; but note $G_1(z_1) = 0$ and $G_2(z_2) = 0$. In this case, the transmission-zero vector corresponding to z_1 is given by $\mathbf{u}_1 = [\alpha, \beta]^\top$, and also $\mathbf{u}_2 = [\beta, -\alpha]^\top$ for the case z_2 ;

$$G(z_1)\mathbf{u}_1 = \begin{bmatrix} 0 & 0 \\ \beta G_2(z_1) & -\alpha G_2(z_1) \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0,$$

$$G(z_2)\mathbf{u}_2 = \begin{bmatrix} \alpha G_1(z_2) & \beta G_1(z_2) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \beta \\ -\alpha \end{bmatrix} = 0.$$

The corresponding eigenvector \mathbf{v}_1 is given, from the proof of Fact 1, by

$$\begin{aligned} \mathbf{v}_1 &= V_1 \mathbf{u}_1 = (z_1 - A)^{-1} C^\dagger S \mathbf{u}_1 \\ &= \begin{bmatrix} z_1 - A_1 & 0 \\ 0 & z_1 - A_2 \end{bmatrix}^{-1} \begin{bmatrix} C_1^* & 0 \\ 0 & C_2^* \end{bmatrix} \\ &\quad \times \begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \\ &= \begin{bmatrix} 1/C_1 \\ 0 \end{bmatrix}, \end{aligned}$$

and also $\mathbf{v}_2 = [0, 1/C_2]^\top$. Note that these are certainly eigenvectors of $-A^\dagger$. Hence the input field state can be prepared to $|\Psi(t_0)\rangle_f = |1_{u'_1}, 0\rangle_f + |0, 1_{u'_2}\rangle_f$ with pulse function

$$\begin{aligned} \mathbf{u}'(t) &= \begin{bmatrix} u'_1(t) \\ u'_2(t) \end{bmatrix} = -x_1 \mathbf{u}_1 e^{z_1 t} - x_2 \mathbf{u}_2 e^{z_2 t} \\ &= -x_1 \begin{bmatrix} \alpha \\ \beta \end{bmatrix} e^{z_1 t} - x_2 \begin{bmatrix} \beta \\ -\alpha \end{bmatrix} e^{z_2 t}, \end{aligned}$$

and the system final state is then given by $|\Psi(0)\rangle_s = v'_1 |1, 0\rangle_s + v'_2 |0, 1\rangle_s$ with $\mathbf{v}' = x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2$. \square

V. SEPARABLE INPUT FIELD

As mentioned before, the input field state (e.g. Eq. (7)) is in general entangled among input channels and is not always easy to generate in experiment. Hence it is reasonable to seek some conditions for perfect state transfer such that the input field state can be prepared relatively easily; in particular here we focus on a *separable state* such as $|1_{e^{zt}}, 0, \dots, 0\rangle_f$.

The first condition is, as expected, that the system has a blocking zero z . In this case, as seen in Eq. (11), $\mathbf{a}^*(0)$ can be represented in terms of z as follows; that is, using the relation $CV = S$ found in the proof of Fact 1, Eq. (4) yields

$$\begin{aligned} V^\top \mathbf{a}^\#(0) &= - \int_{t_0}^0 V^\top e^{-A^\dagger t} C^\top S^\dagger \mathbf{b}^\#(t) dt \\ &= - \int_{t_0}^0 e^{zt} V^\top C^\top S^\dagger \mathbf{b}^\#(t) dt = - \int_{t_0}^0 e^{zt} \mathbf{b}^\#(t) dt \\ &= \mathbf{B}^\#(-e^{zt}) = \begin{bmatrix} B_1^*(-e^{zt}) \\ \vdots \\ B_m^*(-e^{zt}) \end{bmatrix}. \end{aligned} \quad (17)$$

Hence by introducing the normalized pulse function $\zeta(t) = -\sqrt{z + z^*} e^{zt}$, which satisfies $\int_{-\infty}^0 |\zeta(t)|^2 dt = 1$, we have

$$\sqrt{z + z^*} V^\top \mathbf{a}^\#(0) = \mathbf{B}^\#(\zeta).$$

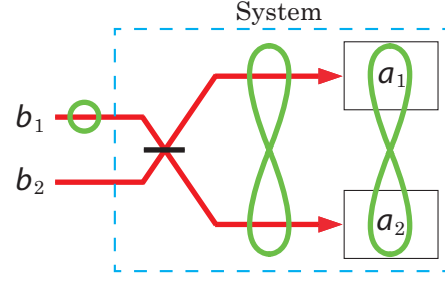


Fig. 2. Schematic for entanglement creation and distribution, which stems from a single photon field state with engineered pulse shape.

A remarkable feature of this relation is that the field operators are “disentangled”, unlike Eqs. (6) and (12) which has the form of entangled operators, $X_1 \otimes I \otimes \dots \otimes I + \dots + I \otimes \dots \otimes I \otimes X_m$. This means that a separable input field state can be chosen. For instance, let us consider

$$|\Psi(t_0)\rangle_f = B_1^*(\zeta) |0, 0, \dots, 0\rangle_f = |1_\zeta, 0, \dots, 0\rangle_f.$$

Then the system final state is given by

$$\begin{aligned} |\Psi(0)\rangle_s &= \mathbf{v}_1^\top \mathbf{a}^\#(t_0) |0, 0, \dots, 0\rangle_s \\ &= v_{1,1} |1, 0, \dots, 0\rangle_s + \dots + v_{1,n} |0, 0, \dots, 1\rangle_s, \end{aligned}$$

where $\mathbf{v}_1 = [v_{1,1}, \dots, v_{1,n}]^\top$ is the first column vector of $\sqrt{z + z^*} V$. That is, if the system has a blocking zero, then a separable field state can be used to achieve the perfect state transfer.

Example 3 (continued from Example 2): If the two subsystems are identical, i.e. $A_1 = A_2$, $C_1 = C_2$, then the two transfer functions become equal, $G_1(s) = G_2(s)$. Hence the whole transfer function matrix is given by

$$G(s) = \begin{bmatrix} \alpha G_1(s) & \beta G_1(s) \\ \beta G_1(s) & -\alpha G_1(s) \end{bmatrix}.$$

Clearly in this case the system has a blocking zero, z , satisfying $G_1(z) = 0$, which is equal to $z = A_1 + |C_1|^2$. Then the V matrix appearing in Eq. (17) is given by

$$\begin{aligned} V &= (z - A)^{-1} C^\dagger S \\ &= \begin{bmatrix} z_1 - A_1 & 0 \\ 0 & z_1 - A_1 \end{bmatrix}^{-1} \begin{bmatrix} C_1^* & 0 \\ 0 & C_1^* \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix} \\ &= \frac{1}{C_1} \begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix}. \end{aligned}$$

Therefore, if we prepare the field initial state as

$$|\Psi(t_0)\rangle_f = B_1^*(\zeta) |0, 0\rangle_f = |1_\zeta, 0\rangle_f,$$

then the system final state is given by

$$|\Psi(0)\rangle_s = \alpha |1, 0\rangle_s + \beta |0, 1\rangle_s,$$

where we have used the fact that $|\sqrt{z + z^*}/C_1| = 1$.

Note again that in this case we only need to prepare a single photon field state living in one channel; then this state becomes entangled after being combined at the beam splitter, and further it is perfectly transferred to the two

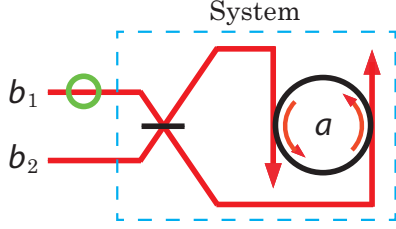


Fig. 3. Single photon absorption into the single-mode ring-resonator with two wave guides.

identical systems, which can be spatially separated as shown in Fig. 2. That is, the schematic proposed here can be used for the purpose of creating and distributing entanglement in a quantum network. Specifically, it can be applied for constructing quantum repeaters [5] to realize a long-distance quantum communication. \square

Another condition such that a separable field input state can be perfectly transferred is as follows; if the transmission-zero vector \mathbf{u} appearing in Eq. (11) is e.g. of the form $\mathbf{u} = [1, 0, \dots, 0]^T$, then Eq. (11) gives

$$\mathbf{v}^\top \mathbf{a}^\#(0) = B_1^*(-e^{zt}),$$

or equivalently $-\sqrt{z+z^*} \mathbf{v}^\top \mathbf{a}^\#(0) = B_1^*(\zeta)$ with the normalized rising exponential function $\zeta(t) = -\sqrt{z+z^*} e^{zt}$. In this case, the separable initial field state $|\Psi(t_0)\rangle_f = |1_\zeta, 0, \dots, 0\rangle_f$ can be perfectly transferred to the system and the final system state is $|\Psi(0)\rangle_s = v'_1 |1^{(1)}\rangle_s + \dots + v'_1 |1^{(n)}\rangle_s$ with v'_j the j th component of the vector $-\sqrt{z+z^*} \mathbf{v}$.

Example 4: Let us consider the system studied in [15], depicted in Fig. 3. The system is a single-mode ring resonator coupled to two optical waveguides, hence it is a 2-input and 2-output system. The wave guides are combined at a beam splitter before connected to the resonator. The transfer function of this system is given by

$$G(s) = \frac{1}{s + (\gamma_1 + \gamma_2)/2} \times \begin{bmatrix} s + (\gamma_2 - \gamma_1)/2 & -\sqrt{\gamma_1 \gamma_2} \\ -\sqrt{\gamma_1 \gamma_2} & s + (\gamma_1 - \gamma_2)/2 \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix},$$

where γ_1 and γ_2 are coupling constants between the resonator and the waveguides. Also α and β are the transmissivity and the reflectivity of the beam splitter, respectively. Clearly $G(s)$ does not have a blocking zero, but (as guaranteed by Fact 3) it has a transmission zero $z = (\gamma_1 + \gamma_2)/2$ with corresponding transmission-zero vector

$$\mathbf{u} = \begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix} \begin{bmatrix} \sqrt{\gamma_1} \\ \sqrt{\gamma_2} \end{bmatrix} = \begin{bmatrix} \alpha\sqrt{\gamma_1} + \beta\sqrt{\gamma_2} \\ \beta\sqrt{\gamma_1} - \alpha\sqrt{\gamma_2} \end{bmatrix}.$$

Therefore from the result of Section IV we need to prepare the following (unnormalized) entangled input field state:

$$|\Psi(t_0)\rangle_f = (\alpha\sqrt{\gamma_1} + \beta\sqrt{\gamma_2})|1_{e^{zt}}, 0\rangle_f + (\beta\sqrt{\gamma_1} - \alpha\sqrt{\gamma_2})|0, 1_{e^{zt}}\rangle_f,$$

to achieve the perfect state transfer. However, in the special case where the parameters satisfy the condition $\beta\sqrt{\gamma_1} - \alpha\sqrt{\gamma_2} = 0$, which leads to $\mathbf{u} = [1, 0]^T$, we only need to prepare a separable input field state $|\Psi(t_0)\rangle_f = |1_{e^{zt}}, 0\rangle_f$, and it is perfectly transferred to the system (see Fig. 3). Note that in this case, because the system is single-mode, the final system state is merely $|1\rangle_s$. \square

VI. CONCLUSION

In this paper, we first showed that the MIMO passive linear system always has a transmission zero, which ensures that a field single-photon state with appropriately engineered pulse function can be perfectly transferred to the system. Although in general the field state has to be an entangled state, under additional specific condition, this requirement can be relaxed; that is, as proven in Section V, a separable field state can be perfectly transferred to the system. This leads to a convenient schematic for creating and distributing entanglement in a quantum network.

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